Final Exam Microeconomics C
Solutions

JANUARY 2012

1. (a) Solve the game below by iterated elimination of strictly dominated strategies. Describe briefly each step.

	X	Y	Z
A	$2,2$	$1,4$	$3,1$
B	$9,3$	$2,4$	$0,3$
C	$3,0$	$0,1$	$0,5$
D	$1,7$	$0,1$	$2,2$

SOLUTION: First, D is dominated by A. Then X is dominated by Y. Then C is dominated by A. Then Z is domnated by Y. Finally A is dominated by B. Thus the surviving strategies are B for player 1 and Y for player 2.

(b) Find all pure and mixed Nash equilibria in the following game:

	t_1	t_2
s_1	$2,1$	$3,0$
s_2	$1,2$	$4,3$
s_3	$0,1$	$0,3$

SOLUTION: There are two pure NE:

$$(s_1, t_1) \text{ and } (s_2, t_2).$$

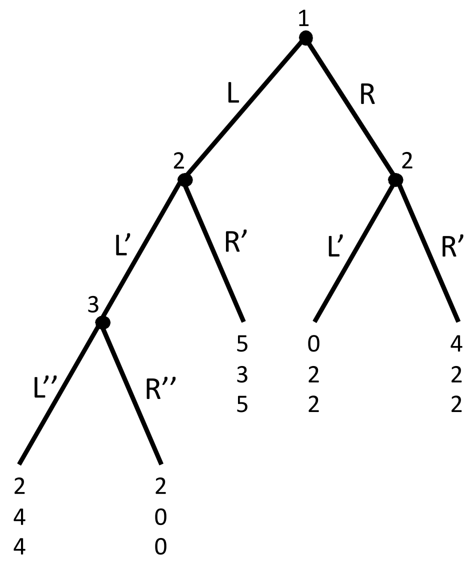
To find the mixed NE, first note that player 1 will never put any probability mass on s_3 since it is strictly dominated by each of his two other strategies. Let p denote the probability that player 1 plays s_1 (then he plays s_2 with probability $1-p$) and let q denote the probability that player 2 plays t_1 (then he plays

t_2 with probability $1 - q$). In a mixed NE each player must be indifferent between all pure strategies that are played with positive probability:

$$\begin{aligned} 2q + 3(1-q) &= q + 4(1-q) \\ 2p + (1-p) &= 3(1-p) \end{aligned}$$

From these two equations we get $p = q = \frac{1}{2}$. Thus there is precisely one (non-pure) mixed NE: Player 1 plays each of the strategies s_1 and s_2 with probability $\frac{1}{2}$ (and strategy s_3 with probability 0). Player 2 plays each of his two strategies (t_1 and t_2) with probability $\frac{1}{2}$.

(c) Consider the game given by the following game tree:



i. Is it a game of perfect or imperfect information? How many subgames are there in the game (excluding the game itself)? What are the possible strategies for the three players?

SOLUTION: There are no information sets with more than one decision node, so it is a game of complete information. There are three subgames in the game: One starting at each of player 2's decision nodes and one starting at player 3's decision node. The strategy sets for the players are:

$$\begin{aligned} \text{Player 1} &: \{L, R\} \\ \text{Player 2} &: \{L'L', L'R', R'L', R'R'\} \\ \text{Player 3} &: \{L'', R''\} \end{aligned}$$

ii. Find all (pure strategy) subgame perfect Nash equilibria.

SOLUTION: Since player 2 is indifferent between L' and R' at his right decision node, there are two pure strategy SPNE:

$$(L, L'L', L'') \text{ and } (R, L'R', L'').$$

iii. Is the strategy profile $(R, L'R', R'')$ a Nash equilibrium?

SOLUTION: Yes, none of the players can achieve a higher payoff by unilaterally deviating to a different strategy. If player 1 devaties to L, then his payoff will be 2 instead of 4. If player 2 devaties to any of his alternative strategies then his payoff will still be 2. If player 3 deviates to L'' then his payoff will still be 2 (because his decision node is never reached given the strategies of player 1 and 2).

2. Three students ($i = 1, 2, 3$) are working on a joint project. The amount of time student i spends on the project is denoted $x_i \geq 0$. The final quality q of the project depends on x_1, x_2, and x_3 in the following way:

$$q(x_1, x_2, x_3) = 2x_1 + 2x_2 + x_3 - x_1x_2 - x_1x_3.$$

Spending time on the project is costly for the students. The cost function for each student is:

$$c_i(x_i) = (x_i)^2.$$

The utility for each student is equal to the final quality of the project minus his cost:

$$u_i(x_1, x_2, x_3) = q(x_1, x_2, x_3) - c_i(x_i).$$

(a) Find the best response functions for the three students. I.e., for each student i, find the optimal amount of time to spend on the project given the time spent by the other two students.

SOLUTION: The maximization problem for student i is:

$$\max_{x_i \geq 0} 2x_1 + 2x_2 + x_3 - x_1x_2 - x_1x_3 - x_i^2.$$

The FOCs for the three students are:

$$\begin{aligned} \text{Student 1} &: \quad 2 - x_2 - x_3 - 2x_1 = 0 \\ \text{Student 2} &: \quad 2 - x_1 - 2x_2 = 0 \\ \text{Student 3} &: \quad 1 - x_1 - 2x_3 = 0 \end{aligned}$$

From these we get the following best response functions:

$$\begin{aligned} x_1 &= \frac{2 - x_2 - x_3}{2} \\ x_2 &= \frac{2 - x_1}{2} \\ x_3 &= \frac{1 - x_1}{2} \end{aligned}$$

(b) Suppose the students simultaneously and independently decide how much time to spend on the project. Find the Nash equilibrium of this game.

SOLUTION: In a NE each student best responds to the time spent by the other two students. Thus, using the results from (a), a NE (x_1^*, x_2^*, x_3^*) must satisfy the following conditions:

$$\begin{aligned} x_1^* &= \frac{2 - x_2^* - x_3^*}{2} \\ x_2^* &= \frac{2 - x_1^*}{2} \\ x_3^* &= \frac{1 - x_1^*}{2} \end{aligned}$$

Solving this system of equations we get:

$$x_1^* = \frac{1}{2},\ x_2^* = \frac{3}{4},\ \text{and } x_3^* = \frac{1}{4}.$$

(c) Now consider the following two stage game: First, student 1 decides how much time to spend on the project. Finally, after observing the choice of student 1, the students 2 and 3 simultaneously and independently decide how much time to spend. Set up the maximization problem facing student 1 in stage one.

SOLUTION: In stage one, student 1 has to take into account how the other two students will respond in the second stage. Since the best response functions of student 2 and 3 depend only on x_1, it immediately follows that they will choose the following amounts of time in stage two:

$$\begin{aligned} x_2 &= \frac{2 - x_1}{2} \\ x_3 &= \frac{1 - x_1}{2} \end{aligned}$$

Thus the maximization problem facing student 1 in stage one is:

$$\max_{x_1 \geq 0} 2x_1 + 2(\frac{2-x_1}{2}) + \frac{1-x_1}{2} - x_1(\frac{2-x_1}{2}) - x_1(\frac{1-x_1}{2}) - x_1^2.$$

By a bit of algebra this problem can be simplified to:

$$\max_{x_i \geq 0} -x_1 + \frac{5}{2}$$

(d) Find the subgame perfect Nash equilibrium of the two stage game from (c). Who works more and who works less than in the Nash equilibrium from (b)? Give an intuitive explanantion.

SOLUTION: By the maximization problem above we immediately get $x_1 = 0$. And then it follows that student 2 and 3 will respond with:

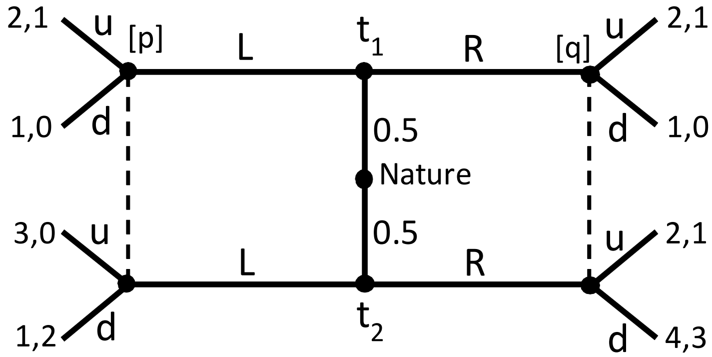
$$\begin{aligned} x_2 &= \frac{2-0}{2} = 1 \\ x_3 &= \frac{1-0}{2} = \frac{1}{2} \end{aligned}$$

Thus the outcome of the SPNE is

$$x_1 = 0,\ x_2 = 1,\text{ and } x_3 = \frac{1}{2}.$$

Thus student 1 works less than in the simultaneous game, student 2 and 3 both work more. In the dynamic game student 1 can commit to spending no time at all on the project, which makes the other two students work more. In the simultaneous game such a commitment is not possible.

3. Consider the following signalling game:



(a) Find a separating perfect Bayesian equilibrium.

SOLUTION: Consider first the sender strategy (L, R), i.e., type t_1 plays L and type t_2 plays R. Then the beliefs of the receiver must be $p = 1$ and $q = 0$. And then it follows that the optimal strategy of the receiver is (u, d), i.e., he plays u after observing the message L and d after observing R. It is then easy to check that the original strategy of the sender is indeed optimal. Thus we have the following separating PBE:

$$[(L, R), (u, d), p = 1, q = 0].$$

Consider then the sender strategy (R, L). Then we must have $p = 0$ and $q = 1$. And then the optimal strategy of the receiver is (d, u). But given this strategy for the receiver, t_2 can profitably deviate from L to R (gives him payoff 2 instead of 1). Thus there does not exist a PBE where the sender's strategy is (R, L).

(b) Find a pooling perfect Bayesian equilibrium. Does it satisfy Signalling Requirement 5 from Gibbons? Does it satisfy Signalling Requirement 6?

SOLUTION: Consider first the sender strategy (L, L). Then we must have $p = \frac{1}{2}$ and then it is optimal for the receiver to play d after observing L. But this means that it is not optimal for t_2 to play L (giving him a payoff of 1) because he can get at least 2 by deviating to R. Thus there does not exist a PBE where the sender strategy is (L, L).

Consider then the sender strategy (R, R). Then we must have $q = \frac{1}{2}$ and then it is optimal for the receiver to play d after observing R. After observing L it is optimal to play d if

$$p \leq 2(1-p) \iff p \leq \frac{2}{3}.$$

It is only optimal for t_1 to play R if the receiver plays d after observing L (it is always optimal for t_2 to play R when the receiver plays d after observing R). Thus we must have $p \leq \frac{2}{3}$. Putting everything together we have that

$$[(R, R), (d, d), p, q = \frac{1}{2}]$$

is a PBE for all $p \leq \frac{2}{3}$ and that there does not exist other PBE where the sender's strategy is (R, R).

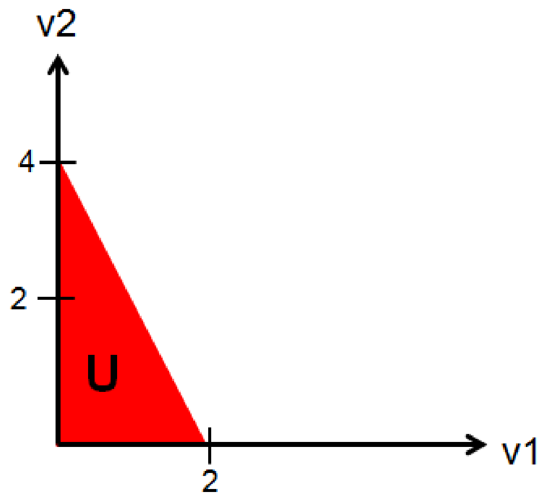
There is no dominated message for either type. Thus all of the pooling PBE from (b) satisfies SR5 (see Gibbons p. 236-7). Then consider SR 6 (see Gibbons p. 239). For any of the pooling PBE, L is equilibrium dominated for t_2 (but not for t_1). Thus SR6 is only satisfied if $1 - p = 0$, which is not true in any of the pooling PBE. Thus none of the pooling PBE satisfies SR6.

4. Consider the bargaining problem (U, d) given by:

$$\begin{aligned} U &= \{(v_1, v_2) | v_1, v_2 \geq 0 \text{ and } v_2 \leq -2v_1 + 4\} \\ d &= (0, 0) \end{aligned}$$

(a) Draw a sketch of U. Can the symmetry axiom (SYM) be used to conclude that the Nash bargaining solution of (U, d) must satisfy $v_1 = v_2$?

SOLUTION: Sketch:



To use the symmetry axiom we must have that U is symmetric:

$$(v_1, v_2) \in U \Rightarrow (v_2, v_1) \in U.$$

This is not the case here (for example $(1,2) \in U$, but $(2,1) \notin U$). Thus we cannot use the symmetry axiom here.

(b) Find the Nash bargaining solution of (U, d).

SOLUTION: The Nash bargaining solution is $(v_1, v_2) = (1, 2)$. This can be shown in several ways, for example the two below:

i. Let

$$\begin{aligned} U' &= \{(2v_1, v_2) | (v_1, v_2) \in U\} \\ d' &= (2d_1, d_2) = (0, 0) \end{aligned}$$

We can use the symmetry axiom on the bargaing problem (U', d') (U' is the triangle with corners at $(0,0)$, $(4,0)$, and $(0,4)$). Thus the Nash bargaining solution of this bargaining problem is $(2,2)$ (follows from symmetry and Pareto efficiency (PAR)). And then we can use the fact that

$$\begin{aligned} U &= \{(\frac{1}{2}v_1, v_2) | (v_1, v_2) \in U'\} \\ d &= (\frac{1}{2}d'_1, d_2) = (0, 0) \end{aligned}$$

and the axiom about invariance of equivalent utility representations (INV) to conclude that the Nash bargaining solution of (U, d) is $(\frac{1}{2}2, 2) = (1, 2)$.

ii. We know that the Nash bargainig solution is given as the solution to

$$\max_{(v_1,v_2)\in U}(v_1 - d_1)(v_2 - d_2).$$

Since $d = (0,0)$ and the solution to this problem obviously must be on the Pareto efficient frontier of U, the maximization problem above can in this case be written

$$\max_{0\leq v_1\leq 2} v_1(-2v_1 + 4).$$

By the FOC we get $v_1 = 1$ and then we get $v_2 = -2(1)+4 = 2$. Thus the Nash bargaining solution is $(1,2)$.